Standard Cosmology Through Similarity

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In this paper we apply dimensional analysis (D.A.) to two cosmological models: Einstein-de Sitter and one Friedmann-Robertson-Walker (FRW) with radiation predominance. We believe that this method leads to the simplest form of solution the differential equations that arise in both models and would be useful as a base for the solution of more complex models. The aim of the paper is therefore rather pedagogical since it tries to show different dimensional techniques.

Key words: FRW Cosmologies, Dimensinal Analysis.

I. INTRODUCTION.

In this paper we try to show how Dimensional Analysis (D.A.) can be formally applied to cosmology. With this aim we shall study two models of known solutions: the Einstein-de Sitter, and one model type Friedmann-Robertson-Walker (FRW) with radiation predominance (see [1]), we shall prove, that we can reach the known results and consequently the usefulness of D.A. as a tool to solve this type of models and eventually those founded on some difficult equations like, for example, alternative theories of the gravitation that envisage the "constants" as scalars functions dependent on time t (see [2]).

Now, we go next to explain step by step the followed dimensional method to obtain a complete solution to the equations that govern each one of the models. We begin in the second section with a brief account of the fundamentals of D.A. Although a more complete understanding is to be found through the advised literature (see [3] and [5]), we hope that such account might suffice for the practical application of D.A. We accompany the explanation with a short amount of examples (that reduce to a single one) for a better understanding. We shall follow a simple scheme. Firstly we shall calculate the multiplicity of the dimensional base to be used for the model under study, choosing one of the possible bases. In second place we shall select the set of quantities and constants to consider in each model since, as stated in third section we are only interested in the fundamental quantities and the unavoidable constants. Then with the help of the Pi theorem we will arrive to the solution of the governing equations of our two models. In the third section the Einstein-de Sitter model will be studied and in the fourth a model type FRW with radiation predominance from our dimensional perspective. In last section the solution is found in function of two monomials related through an unknown function, we will take into account the Barenblatt's criterium (see [5]) to obtain a complete solution to the model. We will justify, from a dimensional point of view, the utilization of the Planck system of units (see [7]), making use of this criterion. In each case we shall contemplate an alternative way to avoid the Barenblatt criterium and solve it completely the problem.

II. DIMENSIONAL ANALYSIS.

In our opinion J. Palacios (see [3]) has been the first author that has tried to arise Dimensional Analysis from a "Method" to a "Theory", where all of its results are derived from a very limited number of postulates (two). Although the theory of Palacios was in our view mostly successful a long time has elapsed and his postulates have been amended by M. Castañs (see [4]) According to Palacios it is possible to select the universal and unavoidable constants as relating two inseparable quantities, "that is, two quantities such that the presence of one of them in a body involves the presence of the other in the same body, so that equal amounts of the first one compounds to equal amounts of the second" one. Palacios, then, postulates (2nd postulate): "unavoidable universal con-

stants are those which relate to inseparable quantities all others are superfluous". The corrections to this postulate by M. Castañs are too slight to be included here as a consequence of the corrected postulate and on account that the constants c, ϵ_0, μ_0 are connected by a well known relation in such a way that the remaining ones in Physics are only five independent constants $G, c, \hbar, k_B$ and either ϵ_0 or μ_0. It is to be observed that $G, c, \hbar, k_B$ are the constants selected by Planck to establish his system of absolute units.

The first postulate reads:
"The fundamental laws can be chosen in such a way that they are relations of proportionality of defined powers of the quantities involved ".

It is, in the opinion of M. Castañs, advantageously replaced by the following statement:

" Dimensional Analysis can be applied to those Physical Theories whose fundamental laws may be written:

$$f(\pi_1, \ldots, \pi_n) = 0 \qquad (1)$$

where π_i they are dimensionless products $(\pi - monomials)$".

Consequences from both postulates (or rather of the second one and the quoted statement) are:

1. It is possible to fix the multiplicity of the dimensional base.

2. The π (Buckingham) theorem may be proved and enunciated without ambiguity.

The recipe to fix the multiplicity of the base is:

"The number of quantities forming the dimensional base (its multiplicity) is given by the difference $m = n - h$ between the total number of quantities (including unavoidable constant) and the rank of the matrix formed with the exponent occurring in the monomial present in the fundamental equations of the concerned theory ."

E_{xample}: **Calculation of multiplicity of dimensional base:** The fundamental equation of thermal radiation is Planck law:

$$\pi_1 = (e^{\pi_2} - 1)^{-1} \qquad (2)$$

where

$$\pi_1 = \frac{u_\gamma c^3}{8\pi h \gamma^3} \qquad \pi_2 = \frac{h\gamma}{k_B T} \qquad (3)$$

Obviously the first postulate is not fulfilled. Instead we use the Castañs statement. The matrix is:

	u_γ	c	h	γ	k_B	T
π_1	1	3	-1	-3	0	0
π_2	0	0	1	1	-1	-1

$$(4)$$

with rank $h = 2$.

Then, the multiplicity is $m = n - h = 6 - 2 = 4$ and we can take a dimensional base of four quantities, for example, $\{L, M, T, \Theta\}$, where Θ means " dimensions of temperature ". Also a base can be formed by $[\gamma] = T^{-1}$ and the constants c, h, k_B . However the same results will be obtained by making use of a base of higher multiplicity (f.e. $\{\gamma, c, h, k_B, M\}$).

The number of independent monomials (dimensionless products) is $i = n - j$ where j is the rank of the matrix of the dimensional exponents of the quantities (and unavoidable constants) relative to a suitable base.

E_{xample}:**A Significant Example:Planck System.** As already mentioned, the system of absolute units proposed by Planck consists of the four universal constants $\{G, c, h, k_B.\}$:

	l_P	G	c	$\hbar$
L	1	3	1	2
M	0	-1	0	1
T	0	-2	-1	-1

$\Longrightarrow \pi_1 = \dfrac{G\hbar}{l_p^2 c^3} \qquad (5)$

	m_P	G	c	$\hbar$
L	0	3	1	2
M	1	-1	0	1
T	0	-2	-1	-1

$\Longrightarrow \pi_2 = \dfrac{\hbar c}{m_p^2 G} \qquad (6)$

	t_P	G	c	$\hbar$
L	0	3	1	2
M	0	-1	0	1
T	1	-2	-1	-1

$\Longrightarrow \pi_3 = \dfrac{G\hbar}{t_p^2 c^5} \qquad (7)$

etc... In the Matrix it is shown how to obtain the mechanical quantities of length, time and mass by using a base L, T, M. We get the monomials π_1, π_2 and π_3. Then:

$$l_p = \sqrt{\frac{G\hbar}{c^3}} \qquad t_p = \sqrt{\frac{G\hbar}{c^5}} \qquad m_p = \sqrt{\frac{\hbar c}{G}} \qquad (8)$$

To obtain Planck's temperature we should consider that its product by k_B has dimensions of energy. This is equivalent to use a dimensional base that includes temperature.

III.   EINSTEIN-DE SITTER.

In this section we shall study the solution of the Einstein-de Sitter model by D.A. We begin showing the essential features of this cosmological model, then we pass to present the field equations that describe it. Departing of these and continuing the traced plan in the previous section we will calculate the multiplicity of the dimensional base and we will choose one of the possible ones. We shall classify the set of quantities and constants and end resolving the equations through dimensional analysis, that is to say, through the Pi theorem.

Our three ingredients of relativistic cosmology are as follows: (we use the standard notation).

1. The cosmological principle which leads to the Robertson-Walker line element,

$$ds^2 = -c^2 dt^2 + f^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right] \tag{9}$$

2. Weyl 's postulate which requires that the substratum is a perfect fluid

$$T_{ij} = (\rho + p)u_i u_j - p g_{ij} \tag{10}$$

3. General relativity

$$R_{ij} - \frac{1}{2}g_{ij}R = \frac{8\pi G}{c^4}T_{ij} \qquad div(T_i^j) = 0 \tag{11}$$

The model is constituted by a perfect fluid with ρ_m, $k = p = 0$. where ρ_m represents the matter density of all galaxies. Under these considerations the field equations are as follows:

$$
\begin{aligned}
(12.1)\quad & 2ff'' + (f')^2 = 0\\
(12.2)\quad & 3(f')^2 = 8\pi G\rho_m f^2\\
(12.3)\quad & f\rho_m' + 3\rho_m f' = 0 \implies \rho_m = M f^{-3}
\end{aligned}
\tag{12}
$$

equation 4.3 is also considered as state equation.

A   Multiplicity of the dimensional base.

We now calculate the multiplicity of the dimensional base. For this purpose we observe that equation (12) can be written (see [3] and [4]):

$$
\begin{aligned}
(12.1)\ & 2\frac{f''}{f} + \left(\frac{f'}{f}\right)^2 = 0 \quad f' = \left[\frac{df}{dt}\right] = \frac{f}{t}\ \ f'' = \left[\frac{d^2f}{dt^2}\right] = \frac{f}{t^2}\\
(12.2)\ & 3\left(\frac{f'}{f}\right)^2 = 8\pi G\rho_m \qquad \left[\left(\frac{f'}{f}\right)^2\right] = t^{-2}\\
(12.3)\ & \rho' + 3\rho\frac{f'}{f} = 0 \qquad \implies \rho_m = \frac{M}{f^3}
\end{aligned}
\tag{13}
$$

These equations can be written in a dimensionally equivalent form

$$
\begin{aligned}
(12.1)\quad & \frac{f}{t^2}\frac{1}{f} + \frac{1}{t^2} = 0\\
(12.2)\quad & \frac{1}{t^2} = G\rho_m \qquad \Rightarrow G^{-1}\rho_m^{-1}t^{-2} = 1\\
(12.3)\quad & \rho_m = \frac{M}{f^3} \qquad \Rightarrow \rho_m f^3 M^{-1} = 1
\end{aligned}
\tag{14}
$$

that leads to the following dimensionless products (see [3] and [4])

$$\implies \begin{cases} \pi_1 := t^{-2}\,G^{-1}\,\rho_m^{-1}\\ \pi_2 := \rho_m\,M^{-1}\,f^{\,3} \end{cases} \tag{15}$$

from the first equation (4.1), we do not obtain dimensional information. We proceed to calculate the multiplicity of the base of this model. The range of the matrix of the exponents of the quantities and constants included in the monomials is 2 as it results immediately from:

$$
\begin{array}{c|ccccc}
 & \rho_m & G & f & t & M\\
\hline
\pi_1 & -1 & -1 & 0 & -2 & 0\\
\pi_2 & 1 & 0 & 3 & 0 & -1
\end{array}
\tag{16}
$$

The multiplicity of the dimensional base is therefore $m = (number\ of\ quantities\ and\ constants) - (range\ of\ the\ matrix)$, in this case it is $m = 3$. Thus we can use the base of classical mechanics $B = \{f, M, t\} \approx \{L, M, T\}$. Other base could be $B' = \{\rho, G, t\}$.

RemarkThe constant c does not appear in the equations (12). This justifies the utilization of Newton's mechanics in the study of the model and simplifies the solution of such equations, that it is reduced to a single $\pi-$monomial.

B   Quantities and constants.

In the case of geometric models, like the previous one, it is trivial the election of the fundamental quantities, since it can be mathematically demonstrated, making use of the Killing's equations that $\{t\}$ is our fundamental quantity. In the case of problems with more physical content we must appeal to our physical knowledge of the problem to be able to choose the set of fundamental quantities (or governing parameters in the nomenclature of Barenblatt). The application of a FRW metric, implies that our universe will be homogeneous and isotropic. This also implies, as it can be proved, that all the quantities that appear in the equations should be functions only on $\{t\}$. That is to say; the radius of the universe f, the energy (matter) density ρ and the expansion speed v are functions on $\{t\}$ and of the unavoidable constants. Therefore, we will say that f, ρ and v are derived quantifies whereas t is the single fundamental quantity that appears in the model. The physical and characteristic constants

are G and M respectively. The rest of the quantities of the model depends on t and on the set of constants that appear in the equations of the model, in this case $\{G, M\}$. The dimensional base of the model is $B = \{L, M, T\}$, and the dimensions of the fundamental quantities and constants are:

1. t cosmic time (fundamental quantity) $[t] = T$

2. G Gravitational constant $[G] = L^3 M^{-1} T^{-2}$

3. M The total mass of the Universe (characteristic constant of the model). $[M] = M$

The dimensional method that we follow consists therefore in: To calculate the multiplicity of the dimensional base, to choose one of the possible bases and to calculate the dimensional equations of each quantity with respect to the elected base. Thus it is needed to make use of our physical and mathematical knowledge of the problem in order to be able to choose the fundamental quantities or governing parameters (in these models the only fundamental quantity that appears is t, but in more complex models, also more fundamental quantities or/and unavoidable universal and characteristic constants might be considered). With these distinctions we shall calculate the derived quantities (f, ρ, v etc...) through the Pi-theorem.

C Solution of the equations through D.A.

Let us calculate through application of the Pi theorem, the radius of the Universe $f(t)$, the velocity of expansion of the galaxies $v(t)$ and the matter density $\rho(t)$ that contains the Universe with radius $f(t)$. Therefore, we shall calculate these quantities in function of $\{G, M, t\}$, that is to say, in function of the unavoidable constants (G, M) and the fundamental quantity, t, by making use of the dimensional base $B = (L, M, T)$

We apply the Pi theorem to obtain:

1 Calculation of the radius of the Universe

The quantities that we will consider are: $f(t) \propto f(G, M, t)$ in the base $B = (L, M, T)$ where the dimensional equation of this quantity is $[f] = L$ with respect to the base B, we get:

$$\begin{array}{c|cccc} & f & G & M & t \\ \hline L & 1 & 3 & 0 & 0 \\ M & 0 & -1 & 1 & 0 \\ T & 0 & -2 & 0 & 1 \end{array} \Rightarrow \pi_3 = \frac{(GM)^{\frac{1}{3}} t^{\frac{2}{3}}}{f(t)} \qquad (17)$$

we have obtained a single monomial that brings us to the following solution:

$$f(t) \propto (GM)^{\frac{1}{3}} t^{\frac{2}{3}} \qquad (18)$$

obtaining the result $f(t) \propto t^{\frac{2}{3}}$. Of course D.A. can not find the value of the numerical and dimensionless constant of proportionality.

We can observe that the differential equation to be solved is:

$$\left(\frac{f'}{f}\right)^2 = \frac{8\pi G}{3} \frac{M}{f^3} \qquad \longmapsto \qquad f(f')^2 = \frac{8\pi GM}{3} \qquad (19)$$

that is immediately integrated after finding an adequate change of variable. Our purpose is to show how the simplest dimensional technique solves immediately the equation. We can explore other possibilities, since if we observe the differential equation (19) we can see that the constants GM always keep the same relation into the equation. If we define a new constant from them $K = GM$ where $[K] = L^3 T^{-2}$, the quantity f can be recalculated but with respect to the next set of fundamental quantities or governing parameters $M = M\{t, K\}$ and $B = \{L, T\}$

$$\begin{array}{c|ccc} & f & K & t \\ \hline L & 1 & 3 & 0 \\ T & 0 & -2 & 1 \end{array} \qquad (20)$$

obtaining a single monomial that brings us again to the above solution.

$$f \propto \left(Kt^2\right)^{1/3} \implies f(t) \propto (GM)^{\frac{1}{3}} t^{\frac{2}{3}} \qquad (21)$$

2 Calculation of the matter density.

The same discussion as above: $\rho_m(t) \propto \rho_m(G, M, t)$. where $[\rho_m] = ML^{-3}$

$$\begin{array}{c|cccc} & \rho_m & G & M & t \\ \hline L & -3 & 3 & 0 & 0 \\ M & 1 & -1 & 1 & 0 \\ T & 0 & -2 & 0 & 1 \end{array} \implies \pi_4 = \frac{1}{\rho_m(t) G t^2} \qquad (22)$$

this single monomial brings us to the following solution:

$$\rho_m(t) \propto \frac{1}{Gt^2} \qquad (23)$$

It is somehow surprising that the result does not depend on M

3 Calculation of the velocity of expansion.

$v(t) \propto v(G, M, t)$ where $[v] = LT^{-1}$

$$\begin{array}{c|cccc} & v & G & M & t \\ \hline L & 1 & 3 & 0 & 0 \\ M & 0 & -1 & 1 & 0 \\ T & -1 & -2 & 0 & 1 \end{array} \Rightarrow \pi_5 = \frac{(GM)^{\frac{1}{3}}}{v(t) t^{\frac{1}{3}}} \qquad (24)$$

$$v(t) \propto (GM)^{\frac{1}{3}} t^{-\frac{1}{3}} \qquad (25)$$

In this model the application of the Pi theorem has carried us to obtain a single dimensionless $\pi-$monomial, for each derived quantity.

IV. FRW WITH RADIATION PREDOMINANCE.

We begin this section continuing the traced plan in section 2, by considering the equations that govern the model. These equations, like the ones of the previous model, are based on the three exposed above basic ingredients, equations $(9, 10, 11)$. We shall select the set of quantities and constants, solving the equations with the help of the Pi theorem.

We will see that, in this case, the solution to be obtained for each one of the calculated quantities depends on certain unknown function φ; $\pi_1 = \varphi(\pi_2)$. To avoid this drawback we will make use of the criterion of Barenblatt. This criterion will eventually enable us to simplify the solution to one of type $\pi_1 = (\pi_2)^n$ being possible to calculate through numerical methods the appropriate value of n. In an alternative way, we shall solve again this model by making use of the Planck system of units simplifying, from a dimensional point of view, the solution obtained through the Baremblatt criterion. We shall end showing how to avoid the use of Barenblatt criterium by considering carefully the differential equations that govern the model. We shall see how to reduce the number of constants and therefore the number of $\pi-monomials$ in such a way that we shall obtain a single one which keeps the problem perfectly solved.

In this second case, a universe with radiation predominance and $(k = 0)$, the equations of Friedmann remain as follows: we maintain the constants to show the dimensional wealth of the equations,

$$\begin{cases} (26.1) & c^2\left(2ff'' + (f')^2\right) = -8\pi G p f^2 \\ (26.2) & 3c^2 (f')^2 = 8\pi G \rho_R f^2 \\ (26.3) & f\rho_R' + 3(p + \rho_R)f' = 0 \Longrightarrow \rho_R f^4 = A \\ (26.4) & \rho_R = a\,\theta^4 \text{ equations of state } \rho_R = \frac{1}{3}p \end{cases} \qquad (26)$$

A   Multiplicity of the dimensional basis.

From the above equations (26) we obtain five $\pi-$monomia. The matrix of the exponents is:

$$\begin{aligned} \pi_6 &:= t^{-2}G^{-1}c^2 p^{-1} \\ \pi_7 &:= t^{-2}G^{-1}c^2 \rho_R^{-1} \\ \pi_8 &:= \rho_R\, a^{-1}\theta^{-4} \\ \pi_9 &:= \rho_R f^4 A^{-1} \\ \pi_{10} &:= \rho_R^{-1}p \end{aligned} \qquad (27)$$

	t	G	c	p	ρ_R	a	f	A	θ
π_6	-2	-1	2	1	0	0	0	0	0
π_7	-2	-1	2	0	1	0	0	0	0
π_8	0	0	0	0	1	-1	0	0	-4
π_9	0	0	0	0	1	0	4	-1	0
π_{10}	0	0	0	1	-1	0	0	0	0

$$(28)$$

that leads us to a multiplicity of the base for this model of 4. A possible base is: $B = \{f, \rho_R, t, \theta\} \approx \{L, M, T, \Theta\}$, where Θ stands for dimension of temperature.

B   Quantities and Constants.

According to what has been stated above, in section (3.2), we consider the following set of quantities and constants in this model, written in the dimensional base $B = \{L, M, T, \Theta\}$

1. t, *cosmic time (fundamental quantity)* $[t] = T$

2. c, *speed of light:* $[c] = LT^{-1}$

3. G, *gravitational constant:* $[G] = L^3 M^{-1} T^{-2}$

4. a, *radiation constant:* $[a] = L^{-1} M^1 T^{-2} \Theta^{-4}$

5. A, *Characteristic constant of the model* $[A] = L^3 M^1 T^{-2}$

Therefore, by using the dimensional base B the derived quantities will appear in function of the unavoidable constants (G, c, a, A) and the fundamental quantity, t.

C   Solution through D.A.

We would like to obtain expressions for the temperature θ, the energy density ρ_R, the radius of the universe $f(t)$ (the latter quantity is fundamental since it determines the metric and therefore the geometry of our space-time) and finally the entropy s and entropy density S.

In this section we shall take into account the Barenblatt's criterium ([5]) that eventually will allow solutions of the type $\pi_i = (\pi_j)^n$.

1   Calculation of the temperature.

We go next to carry out the calculation by three methods. The first one through dimensional analysis i.e. by writing the matrix of the exponents and applying the Pi theorem, a second one, also dimensional,

by using the *Planck's system of units.* In both methods we must take into account the **Barenblatt's criterium** to arrive to the complete solution of the problem. We end showing an alternative way that brings us to a complete solution of the equation avoiding in such a way the Barenblatt criterium, since the use of numerical data is always is unsafe.

We assume $\theta \propto \theta(G, c, A, a, t)$,where we designate for dimensions of θ is $[\theta] = \Theta$

$$
\begin{array}{c|cccccc}
 & \theta & G & c & A & a & t \\
\hline
L & 0 & 3 & 1 & 3 & -1 & 0 \\
M & 0 & -1 & 0 & 1 & 1 & 0 \\
T & 0 & -2 & -1 & -2 & -2 & 1 \\
\Theta & 1 & 0 & 0 & 0 & -4 & 0
\end{array}
\tag{29}
$$

We obtain 2 dimensionless $\pi-$monomia:

$$
\pi_{11} = \frac{\theta c a^{\frac{1}{4}} t}{A^{\frac{1}{4}}} \qquad \pi_{12} = \frac{GA}{c^6 t^2}
\tag{30}
$$

The solution that classic D.A. gives us is:

$$
\pi_{11} = \varphi(\pi_{12}) \qquad \Longrightarrow \qquad \theta = \frac{A^{\frac{1}{4}}}{c a^{\frac{1}{4}} t} \varphi\left(\frac{GA}{c^6 t^2}\right)
\tag{31}
$$

where φ represents an unknown function.

If we take into account the Barenblatt's criterium ([5]), then we can suppose that the solution is of the form $\pi_{11} = (\pi_{12})^n$. For this, we need to know the orders of magnitude of each one of the $\pi-$monomials ([9]):

$$
\pi_{11} = \frac{\theta_0 c a^{\frac{1}{4}} t_0}{A^{\frac{1}{4}}} \approx 10^{2.614} \qquad \pi_{12} = \frac{GA}{c^6 t_0^2} \approx 10^{-10.457}
\tag{32}
$$

indicating us that the solution can be expressed as:

$$
\pi_{11} = (\pi_{12})^n \qquad \Longrightarrow \qquad \theta \propto \frac{A^{\frac{1}{4}}}{c a^{\frac{1}{4}} t} \left(\frac{GA}{c^6 t^2}\right)^n
\tag{33}
$$

$$
n = \frac{\log \pi_{11}}{\log \pi_{12}} = \frac{2.614}{-10.457} = -0.250 \cong -\frac{1}{4}
\tag{34}
$$

Then we have obtained through numerical calculation $n \simeq -\frac{1}{4}$. The final result coincides with the theoretical one except for a numerical factor.

$$
\theta(t) \propto \left(\frac{c^2}{Ga}\right)^{\frac{1}{4}} \cdot t^{-\frac{1}{2}} \qquad \theta(t) = \left(\frac{3c^2}{32\pi Ga}\right)^{\frac{1}{4}} \cdot t^{-\frac{1}{2}}
\tag{35}
$$

Now we are going to use the **Planck's system of units**, and Barenblatt's criterium. Since we know that all the quantities depend only on t , then we can suppose that the temperature will be given by a dimensionless product involving Planck's temperature, Planck's time and the cosmic time. The solution is evidently:

$$
\theta(t) \propto \theta_p \cdot \varphi\left(\frac{t_p}{t}\right)
\tag{36}
$$

If we take into account the Barenblatt's criterium, we may suppose that the solution will be of the form:

$$
\Longrightarrow \qquad \theta(t) \propto \theta_p \cdot \left(\frac{t_p}{t}\right)^n
\tag{37}
$$

since

$$
\pi_{13} = \left(\frac{\theta_0}{\theta_p}\right) \approx 10^{-31.715} \qquad \pi_{14} = \left(\frac{t_p}{t_0}\right) \approx 10^{-63.522}
\tag{38}
$$

obtaining the value of n through a simple numerical calculation.

$$
n = \left(\frac{\log \pi_{13}}{\log \pi_{14}}\right) = \frac{-31.715}{-63.522} = 0.499 \approx \frac{1}{2}
\tag{39}
$$

The solution that we obtain is therefore:

$$
\theta(t) \propto \theta_p \cdot \left(\frac{t_p}{t}\right)^{\frac{1}{2}}
\tag{40}
$$

This is another dimensional solution. Evidently, both expressions coincide (compare (40) after simplifying it ([10]) with (35)).

We can explore other possibilities. By examining the relation between the constants. For example always in our formulas the quotient $\frac{G}{c^2}$ holds . For this reason we define a single new constant $B = \frac{G}{c^2}$ where $[B] = LM^{-1}$. Whit this new constant the set of fundamental quantities results: $M = M(B, A, t)$ while the dimensional base still holds. This new consideration brings us to obtain the following matrix:

$$
\begin{array}{c|ccccc}
 & \theta & B & A & a & t \\
\hline
L & 0 & 1 & 3 & -1 & 0 \\
M & 0 & -1 & 1 & 1 & 0 \\
T & 0 & 0 & -2 & -2 & 1 \\
\Theta & 1 & 0 & 0 & -4 & 0
\end{array}
\tag{41}
$$

obtaining a single monomial that brings us to the following solution

$$
\theta(t) \propto \left(\frac{1}{Bat^2}\right)^{\frac{1}{4}} = \left(\frac{c^2}{Ga}\right)^{\frac{1}{4}} t^{-\frac{1}{2}}
\tag{42}
$$

This method allows to avoid Barenblatt´s criterium, always very uncertain, since it depends of observations. Evidently all this considerations could be made in the cases bellow.

2 Calculation of energy density.

$\rho_R \propto \rho(G, c, A, t)$ where $[\rho_R] = L^{-1}MT^{-2}\Theta^0$. The dimensional equation of this quantity can not depend of the constant a since the resting quantities and constants are independent of temperature. We have them:

$$
\begin{array}{c|ccccc}
 & \rho_R & G & c & A & t \\
\hline
L & -1 & 3 & 1 & 3 & 0 \\
M & 1 & -1 & 0 & 1 & 0 \\
T & -2 & -2 & -1 & -2 & 1
\end{array}
\tag{43}
$$

obtaining:

$$
\pi_{15} = \frac{A}{c^4 t^4 \rho_R} \qquad \pi_{12} = \frac{GA}{c^6 t^2}
\tag{44}
$$

that lead us to:

$$
\rho_R \propto \frac{A}{c^4 t^4} \cdot \varphi \left(\frac{GA}{c^6 t^2} \right)
\tag{45}
$$

In this case we can not apply the Baremblatt's criterium, since the absolute values of the orders of magnitude coincide:

$$
\pi_{15} = \frac{c^4 t_0^4 \rho_{R_0}}{A} \approx 10^{10.457} \qquad \pi_{12} = \frac{GA}{c^6 t_0^2} \approx 10^{-10.457}
\tag{46}
$$

however if we insist on assuming that the solution will be of the form $\pi_{15} = (\pi_{12})^n$. Obviously $n = -1$ being admittedly

$$
\rho_R \propto \frac{c^2}{Gt^2} \qquad\qquad \rho_R = \frac{3c^2}{32\pi G t^2}
\tag{47}
$$

Using the Planck's system:

$$
\pi_{16} = \left(\frac{\rho_{R_0}}{\rho_p} \right) \approx 10^{-127.045} \qquad \pi_{14} = \left(\frac{t_p}{t_0} \right) \approx 10^{-63.522}
\tag{48}
$$

we see after comparing the two $\pi-$monomials that we can apply the Barenblatt's criterium and we suppose that the solution has the form:

$$
\rho_R(t) \propto \rho_p \cdot \left(\frac{t_p}{t} \right)^n
\tag{49}
$$

$$
n \approx \left(\frac{\log \pi_{16}}{\log \pi_{14}} \right) = \frac{-127.045}{-63.522} = 1.99 \approx 2
\tag{50}
$$

whit $n = 2$. Both expressions (compare (49) with (47)) coincide.

We obtain the same solution by another approach (as Dirac in his LNH) comparing the $\pi-$monomials. Since their orders of magnitude coincide we can write

$$
\pi'_{15} = \frac{A}{c^4 t_0^4 \rho_{R_0}} \approx 10^{-10.457} \qquad \pi_{12} = \frac{GA}{c^6 t_0^2} \approx 10^{-10.457}
\tag{51}
$$

$$
\pi'_{15} = \frac{A}{c^4 t^4 \rho_R} = \frac{GA}{c^6 t^2} = \pi_{12} \Longrightarrow \rho_R \propto \frac{c^2}{Gt^2}
\tag{52}
$$

We can avoid the use of Baremblatt criterium if as in the section above we consider the trick of using the new constant B. In this case, $\rho = \rho(A, B, t)$ such approach brings us to the following solution:

$$
\rho \propto \frac{1}{Bt^2} \qquad \rho_R \propto \frac{c^2}{Gt^2}
\tag{53}
$$

observing again that this tactic is correct.

3 Calculation of the radius of the universe

This quantity $f(t)$ depends on (G, c, A, t), where $[f] = L$. Therefore now :

$$
\begin{array}{c|ccccc}
 & f & G & c & A & t \\
\hline
L & 1 & 3 & 1 & 3 & 0 \\
M & 0 & -1 & 0 & 1 & 0 \\
T & 0 & -2 & -1 & -2 & 1
\end{array}
\tag{54}
$$

and we obtain two dimensionless products. The solution is:

$$
\pi_{17} = \frac{f}{ct} \qquad \pi_{12} = \frac{GA}{c^6 t^2} \qquad \Longrightarrow f \propto ct \cdot \varphi \left(\frac{GA}{c^6 t^2} \right)
\tag{55}
$$

where the orders of magnitude are:

$$
\pi_{17} = \frac{f_0}{ct_0} \approx 10^{-2.614} \qquad \pi_{12} = \frac{GA}{c^6 t_0^2} \approx 10^{-10.457}
\tag{56}
$$

this situation coincides with the paragraph 4.3.2 (Calculation of the temperature)

$$
f \propto ct \cdot \left(\frac{GA}{c^6 t^2} \right)^n
\tag{57}
$$

i.e. the Barenblatt's criterium enables us to take a solution of the type: $\pi_{17} = (\pi_{12})^n$. Then we calculate n as:

$$
\pi_{17} = (\pi_{12})^n \qquad \Rightarrow \qquad n \approx \left(\frac{\log \pi_1}{\log \pi_2} \right) \approx \frac{1}{4}
\tag{58}
$$

and we can write the following expression:

$$
f(t) \propto \left(\frac{GA}{c^2} \right)^{\frac{1}{4}} t^{\frac{1}{2}} \qquad\qquad f(t) = \left(\frac{32\pi GA}{3c^2} \right)^{\frac{1}{4}} t^{\frac{1}{2}}
\tag{59}
$$

Also the orders of magnitude in absolute value of π_{17} given by (55) and π_{11} from (29) are the same (this is related to the hypothesis LNH of Dirac) proving that today:

$$
\pi'_{17} = \frac{ct_0}{f_0} \approx 10^{2.614} \qquad \pi_{11} = \frac{\theta_0 c a^{\frac{1}{4}} t_0}{A^{\frac{1}{4}}} \approx 10^{2.614}
\tag{60}
$$

$$\frac{ct}{f} = \frac{\theta c a^{\frac{1}{4}} t}{A^{\frac{1}{4}}} \Longrightarrow f = \frac{A^{\frac{1}{4}}}{\theta a^{\frac{1}{4}}} \qquad \Longleftrightarrow \rho_R f^4 = A \quad (61)$$

this solution verifies the equations (if we substitute $\theta a^{\frac{1}{4}}$ by its value calculated in equation (35) we obtain (59) as a result).

The differential equation to be solved is now:

$$f^2 (f')^2 = \frac{8\pi G}{3c^2} A \qquad (62)$$

through D.A. we can integrate it easily obtaining the solution (59). We observe that the governing parameters in this case are $M = M(t, G, c, A)$ and if we do the same trick as before we can express the relation $\frac{G}{c^2} = B$, obtaining a single $\pi - monomia$ avoiding the Baremblatt criterium. We can increase the simplicity of the problem if we do $\frac{GA}{c^2} = N$ where $[N] = L^4 T^{-2}$ and using a simple dimensional base $B' = \{L, T\}$. In this case the solution is:

$$f \propto N^{1/4} t^{1/2} \qquad (63)$$

obviously if we simplify N for its valor it is observed that we recover the classical solution

$$f \propto N^{1/4} t^{1/2} \qquad f \propto \left(\frac{GA}{c^2}\right)^{1/4} t^{1/2} \qquad (64)$$

In the calculation of this quantity we cannot use the Planck's system of units as before. This question is known as the Planck problem and it was Zeldovich who pointed out this conflict in the standard model. Zeldovich (see [8]) emphasized that this is perhaps the most fundamental and serious problem of the standard cosmology. This mismatch of scales is generally referred to as the Planck problem.

4   Calculation of the Entropy.

Equations (26) correspond to no variation of entropy in the Universe. We mean to calculate the entropy of this Universe, s, and the entropy density, S.

For its calculation we follow the same method, but in this case we do not consider t since we know that this quantity is constant ([1]) consequently:
$s \propto s(G, c, A, a)$ where $[s] = L^2 M T^{-2} \Theta^{-1}$ and we get:

	s	G	c	A	a
L	2	3	1	3	−1
M	1	−1	0	1	1
T	−2	−2	−1	−2	−2
Θ	−1	0	0	0	−4

(65)

$$\Longrightarrow s \propto \left(A^3 a\right)^{\frac{1}{4}} \qquad (66)$$

This value is too high, being a difficult issue of justifying within the model.

We suppose that we ignore the behavior of this quantity (we do not know that $s = const.$). In this case we could follow up the exposed method up to now, that is to say, we would calculate this quantity in function of (t, G, c, a, A). In this case the utilization of the constant a is necessary for dimensional considerations. As we have seen, we obtained two monomials: $\pi_{18} = \left(\frac{s}{(A^3 a)^{\frac{1}{4}}}\right)$ and $\pi_{12} = \left(\frac{GA}{c^6 t^2}\right)$. Then the solution that we would obtain would be. $\pi_{18} = \varphi(\pi_{12})$. Taking into account the criterion of Baremblatt and knowing that: $s \approx 10^{64.18735} J K^{-1}$ we get, $\pi_{18} \approx 10^0$, this enables us to write $\pi_{18} = (\pi_{12})^n$ iif $n = 0$ obtaining therefore the solution that gives us our first position.

As above, we can explore the possibility of reducing the number of constants. In this case such reduction brings us to the following solution:

$$s = s(A, B, t) \qquad \Longrightarrow s \propto \left(A^3 a\right)^{\frac{1}{4}} \qquad (67)$$

For the calculation of $S \propto S(G, c, A, a, t)$

	S	G	c	A	a	t
L	−1	3	1	3	−1	0
M	1	−1	0	1	1	0
T	−2	−2	−1	−2	−2	1
Θ	−1	0	0	0	−4	0

(68)

$$S \propto \left(\frac{a^{\frac{1}{4}} A^{\frac{3}{4}}}{c^3 t^3}\right) \cdot \varphi\left(\frac{GA}{c^6 t^2}\right) \qquad (69)$$

as we do not know here the numerical values of such quantities we can not operate as before. But we know that $S = s/f^3$. Simplifying both expressions and eliminating a with $\left(a \propto \frac{k_B}{c^3 \hbar^3}\right)$ we obtain

$$S \propto \left(\frac{a^{\frac{1}{4}} c^{\frac{6}{4}}}{G^{\frac{3}{4}} t^{\frac{3}{2}}}\right) \propto \left(\frac{ac^6}{G^3 t^6}\right)^{\frac{1}{4}} \propto k_B \left(\frac{c}{G\hbar t^2}\right)^{\frac{3}{4}} \qquad (70)$$

Observe that $S \propto a\theta^3$. If we simplify this expression then we obtain the results above (70). If we reduce the number of constants, this tactic brings us to obtain a single monomial:

$$S \propto \left(\frac{a}{B^3 t^6}\right)^{\frac{1}{4}} \qquad \Longrightarrow S \propto \left(\frac{ac^6}{G^3 t^6}\right)^{\frac{1}{4}} \qquad (71)$$

V.   CONCLUSIONS.

We have shown formally the fruitful application of the D.A. to these two cosmological concrete models.

For the case of the Einstein-de Sitter one, we have arrived at the solution in a trivial way since it appears a single dimensionless product, whereas for the model type FRW with radiation predominance we have explored various possibilities in particular we have taken into account the criterion of Barenblatt in order to obtain the complete solution of the equations. For the latter model we have justified, from a dimensional point of view, the utilization of the Planck's system of units. We believe therefore that the method developed here can be useful for solving more complex models whose equations might be of difficult integration. We try to be rigorous when formalizing all the required steps. But if one has got a good knowledge (from the physical point of view) of the behavior of the model one does not need be so scrupulous. There is no need of developing the equations to obtain relationships between the quantities that form part of the model. We think therefore that this method may also have some pedagogical interest.

VI. TABLE OF QUANTITIES AND CONSTANTS.

Quantity I.S.	Constant I.S.
$\theta_0 \approx 10^{0.436} K$	$G \approx 10^{-10.1757} m^3 kg^{-1} s^{-2}$
$f_0 \approx 10^{26} m$	$c \approx 10^{8.476821} ms^{-1}$
$t_0 \approx 10^{20.252} s$	$a \approx 10^{-15.121153} Jm^{-3} K^{-4}$
$\rho_{R_0} \approx 10^{-13.379} Jm^{-3}$	$A \approx 10^{90.62}\ m^3 kgs^{-2}$

Where θ_0 represents the temperature of background cosmic microwave radiation today i.e. $\theta_0 \approx 2.73K \Rightarrow \log_{10}(2.73) = 0.436162 \Rightarrow \theta_0 \approx 10^{0.436} K$ and f_0 is the radius of the Universe, t_0 represents the approximate age of the Universe and ρ_{R_0} is the energy density of the radiation today

Planck system: length, time, mass, energy density and temperature.

Quantity	Definition	N. value I.S.
l_p	$\sqrt{\frac{G\hbar}{c^3}}$	$10^{-34.7915} m$
t_p	$\sqrt{\frac{G\hbar}{c^5}}$	$10^{-43.2684} s$
m_p	$\sqrt{\frac{\hbar c}{G}}$	$10^{-7.6622}\ kg$
ρ_p	$\frac{m_p c^2}{l_p^3}$	$10^{113.666} Jm^{-3}$
θ_p	$\sqrt{\frac{\hbar c^5}{k_B^2 G}}$	$10^{32.1514} K$

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[9] See in section VI the table I of numerical values

[10] See table II in section 6